

On the $\sqrt{2}$ puzzle in $\text{AdS}_2/\text{CFT}_1$

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Abstract

In this letter we analyze the Hamiltonian formulation of the Jackiw-Teitelboim model of 2D gravity and calculate the central charge associated with the asymptotic symmetries, taking care of boundary terms. For black hole solutions, we show that there is no $\sqrt{2}$ discrepancy between the thermodynamical entropy and the statistical one obtained via Cardy's formula.

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Two-dimensional dilatonic models of gravity received recently much attention. Their importance relies on their role in the conjectured AdS/CFT correspondence [1, 2] and in describing some properties of higher-dimensional black-holes (see e.g. [3]). Two distinct realizations of the $\text{AdS}_2/\text{CFT}_1$ correspondence are known [4]. The simplest one is known as $\text{AdS}_2/\text{CFT}_1$ and can be understood in terms of boundary fields [3], but is believed to be plagued by a value of the central charge that leads to a mismatch between statistical and thermodynamical entropy of black hole solutions of the Jackiw-Teitelboim model. In a previous paper [5] we have shown that if the $U(1)$ timelike symmetry of a stationary solutions has fixed points, then the hamiltonian for gravity has a boundary term coming from the fixed point set, and moreover that this term can be interpreted in a CFT language. This gave a statistical entropy matching correctly to the usual thermodynamics of black holes. It was clear from [5] that this horizon boundary term is a general feature of gravity in any dimension.

Take for example the BTZ[6] black hole, with metric

$$ds^2 = - \left(-8mG_3 + \frac{r^2}{\ell^2} \right) dt^2 + \left(-8mG_3 + \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 d\phi^2 \quad (1)$$

We normalize the surface gravity to one, so $\sqrt{8mG_3} = \ell$ and the euclidean metric describes a disk times a circle, both with periodicity 2π . The mass, i.e. the boundary term at infinity, is then $m = \ell^2/8G_3$ and the boundary term at the horizon is evaluated as $\ell^2/4G_3$. As explained in [5], this would give a central charge $c = 3\ell^2/G_3$, $L_0 = m$ and the entropy takes the value

$$S = 2\pi \sqrt{\frac{cL_0}{6}} = \frac{\pi\ell^2}{2G_3} = \frac{A}{4} \quad (2)$$

matching the Bekenstein-Hawking's entropy. On the other hand, Strominger[7] had already found the correct entropy by using the central charge associated with the asymptotic symmetries in AdS_3 at infinity, as given by Brown-Henneaux[8] years ago. Note however that Brown-Henneaux's central charge is different from our $c = 3r_+/G_3$, which depends from the mass of

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the black hole. This would mean that the central charge actually depends on the states on which the Virasoro operators act. The coincidence of the present calculation with that of Strominger suggest that the boundary CFT has enough information to describe the black holes in the bulk [9], certainly an aspect of the AdS_3/CFT_2 correspondence.

The aim of this paper is to show that the same happens to the central charge associated with the asymptotic symmetries of AdS_2 , that has indeed the value needed to account for black hole entropy.

As a starting point let's consider the JT action¹

$$S = \frac{1}{2} \int d^2x \sqrt{-g} \eta [R + 2\lambda^2] \quad (3)$$

The general stationary solution to the equations of motion are given by

$$ds^2 = -(\lambda^2 x^2 - a^2) dt^2 + (\lambda^2 x^2 - a^2)^{-1} dx^2 \quad (4)$$

for the metric and

$$\eta = \eta_0 \lambda x \quad (5)$$

for the dilaton. For positive a^2 , these solutions can be interpreted as black holes of mass

$$M = \frac{1}{2} \eta_0 \lambda a^2 \quad (6)$$

and entropy

$$S = 2\pi \eta_0 a \quad (7)$$

The metric is defined to be asymptotically AdS_2 if, for $x \rightarrow \infty$

$$\begin{aligned} g_{tt} &= -\lambda^2 x^2 + \gamma_{tt}(t) + O\left(\frac{1}{x^2}\right) \\ g_{tx} &= \frac{\gamma_{tx}(t)}{\lambda^3 x^3} + O\left(\frac{1}{x^5}\right) \\ g_{xx} &= \frac{1}{\lambda^2 x^2} + \frac{\gamma_{xx}(t)}{\lambda^4 x^4} + O\left(\frac{1}{x^6}\right) \end{aligned}$$

while the asymptotic behaviour of the dilaton is taken to be

$$\eta = \eta_0 \left(\lambda \rho(t) x + \frac{\gamma_{\phi\phi}}{2\lambda x} \right) + O\left(\frac{1}{x^3}\right)$$

The boundary fields are then $\gamma_{tt}, \gamma_{tx}, \gamma_{xx}, \gamma_{\phi\phi}, \rho$ and they transform as conformal fields under the action of the asymptotic symmetries. The task is to calculate the charge J associated with these symmetries and this can be easily done in the Hamiltonian formalism.

Hamiltonian formulation of 2D dilatonic gravity theories has been already carried out, e.g. in [10, 11], and other authors usually refer to that analysis. The starting point is the ADM decomposition of the metric

$$ds^2 = -N^2 dt^2 + \sigma^2 (dx + N^x dt)^2$$

which leads to an Hamiltonian of the following form

$$H_{bulk} = \int dx (N\mathcal{H} + N^x \mathcal{H}_x) \quad (8)$$

¹see [1, 2, 3] for references, definitions and notation. We try to summarize the points needed to make our discussion self-contained

As explicitly stated in [10], in this expression some boundary terms have been dropped. We already noticed in [5] that the full Hamiltonian must include this boundary terms and hence it reads

$$H_{tot} = H_{bulk} + \lim_{x \rightarrow \infty} \left[\left(\frac{\eta N'}{\sigma} - \eta \Pi_\eta N^x \right) + \left(\sigma \Pi_\sigma N^x - \frac{\eta' N}{\sigma} \right) \right] \quad (9)$$

where a prime denotes differentiation w.r.t. x and the two Π 's are the canonically conjugate momenta to the fields in the subscript.

The two boundary terms have different origins. The first one comes from an integration by parts performed in the action integral to get rid of second order derivatives; the second one arises as a consequence of a second integration by parts performed after the Legendre transformation in order to write the Hamiltonian in the form shown above, with N and N^x acting as Lagrange multipliers.

It is usually believed that there's no need to keep track of these boundary terms, since we should be able to recover them by requiring the final Hamiltonian to be differentiable. Using this approach, the boundary term that has to be added to the variation of (8) is

$$\delta J = - \lim_{x \rightarrow \infty} [N(\sigma^{-1} \delta \eta' - \sigma^{-2} \eta' \delta \sigma) - N'(\sigma^{-1} \delta \eta) + N^x(\Pi_\eta \delta \eta - \sigma \delta \Pi_\sigma)]$$

or using the boundary fields

$$\delta J[\epsilon] = \eta_0 \left[\lambda \epsilon (\gamma_{tt} \delta \rho + \frac{\rho}{2} \delta \gamma_{xx} + \delta \gamma_{\phi\phi}) + \frac{\dot{\epsilon} \delta \rho}{\lambda} - \frac{\ddot{\epsilon} \delta \rho}{\lambda} \right] \quad (10)$$

where $\epsilon(t)$ is the function of time which enters into the definition of the asymptotic symmetries and a dot denotes differentiation w.r.t. time. Moreover in [2] it's claimed that the equation of motion of the dilaton (the 00 and 01 component respectively) gives the constraints

$$\lambda^{-2} \ddot{\rho} = \rho(\gamma_{tt} - \gamma_{xx}) - \gamma_{\phi\phi} \quad (11)$$

$$\dot{\rho} \gamma_{tt} + \frac{\rho}{2} \dot{\gamma}_{xx} + \dot{\gamma}_{\phi\phi} = 0 \quad (12)$$

Using these equations and expanding near the classical solution $\rho = 1 + \bar{\rho}$, their value for the charge for on-shell field configurations, up to a constant which equals the mass (6) and up to a total time derivative (which plays no role, since the actual charge is obtained by integrating the above expression over time [1]), is

$$J[\epsilon] = -2\eta_0/\lambda \epsilon \ddot{\rho} + \epsilon M \quad (13)$$

In the spirit of AdS/CFT correspondence, this charge (once we drop the mass term) is interpreted as a conformal stress-tensor, whose anomaly in the conformal transformation gives the central charge of the associated Virasoro algebra. The value of the central charge obtained from (13) is claimed to be $C = 24\eta_0$. This calculation has two shortcomings. First, in obtaining (10) the boundary terms have been discarded. Second, the use of (12) is controversial: equation (11) must definitively hold, since it's obtained by requiring the vanishing of the $O(x)$ term in the 00 component of the dilaton equation of motion; equation (12), on the other hand, comes from the leading term of $O(1/x^2)$ in the 01 component, which therefore becomes a null identity as x goes to infinity. Third, to obtain a Virasoro's algebra the mass term has to be dropped, but setting $M = 0$ (i.e. $L_0 = 0$) Cardy's formula cannot be used, since it holds if $L_0 \gg C$.

The boundary term in (the phase space functional) H_{tot} (9) can be rewritten as

$$\eta_0 [\lambda \epsilon (\gamma_{tt} \rho + \gamma_{\phi\phi}) + \dot{\epsilon} \rho / \lambda - \rho \ddot{\epsilon} / \lambda]$$

Including this term in the variation leads to a modified expression for δJ

$$\delta J_{tot}[\epsilon] = \eta_0 \lambda \epsilon \rho [\delta \gamma_{tt} - \delta \gamma_{xx} / 2]$$

This can be rewritten in the more useful form

$$\delta J_{tot}[\epsilon] = \eta_0 \lambda \epsilon \{ \delta[\rho(\gamma_{tt} - \gamma_{xx})] + \delta\gamma_{xx}/2 - \delta\rho(\gamma_{tt} - \gamma_{xx}) \} \quad (14)$$

In this form, we can immediately insert (11). Considering as above configurations near the classical one it's easy to get

$$J_{tot} = \eta_0/\lambda \epsilon \ddot{\rho} + \epsilon M \quad (15)$$

This gives a conformal stress-tensor exactly 1/2 of the previous result (13), modulo a sign, and we expect the central charge also to get a 1/2 factor: $\mathcal{C} = 12\eta_0$.

To verify this, we calculate the variation of the stress-tensor (keeping the mass term!); using (14), (11) and the transformation laws for the fields given in [2], we get near the classical solutions:

$$\epsilon \delta_\omega \Theta = \epsilon (\omega \dot{\Theta} + 2\dot{\omega} \Theta) - \frac{\eta_0}{\lambda} \epsilon \ddot{\omega}$$

and this equals the Dirac bracket between the charges. Using the Fourier expansions:

$$\epsilon = \frac{1}{\lambda} \sum_m a_m e^{i\lambda m t}, \quad \omega = \frac{1}{\lambda} \sum_n a_n e^{i\lambda n t}, \quad \Theta = \lambda \sum_k L_k e^{-i\lambda k t}$$

we get a (classical) Virasoro algebra

$$[L_m, L_n] = -i\{(m-n)L_{m+n} + \eta_0 m^3 \delta_{m+n}\}$$

Hence it's shown that the (quantum) central charge is $\mathcal{C} = 12\eta_0$. With one qualification, this is the same result as obtained in [3]. The qualification to be added is that the authors of [3] refer to a non scalar dilaton, here we improve upon the scalar dilaton theory studied in [2].

Turning to the issue of the statistical entropy, it is given by Cardy's formula, provided that $a^2 \gg 24$,

$$S_{stat} = 2\pi \sqrt{\mathcal{C} L_0 / 6}$$

where $L_0 = M/\lambda$. Substituting for the value of the central charge

$$S_{stat} = 2\pi \eta_0 a$$

which agrees with the thermodynamical expression (7). This demonstrates also that, at least for 2D systems, discarding boundary terms leads to wrong conclusions. This is probably related to the topological nature of 2D gravity. As a remark, for higher dimensional AdS black holes³, the horizon central charge (calculated à la Carlip[19]) matches the Bekenstein-Hawking entropy provided the mass of the black hole, but not the mass as measured from infinity, be directly related to the Virasoro's generator L_0 . On the other hand, the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence shows that the entropy scales with the area of the event horizon, so the CFT at infinity with a black hole in the bulk must count correctly the number of degrees of freedom of the horizon (see [20] for a discussion of density of states in CFT), in agreement with Witten's argument [21].

Our result removes one of the problems in the $\text{AdS}_2/\text{CFT}_1$ scenario but also introduces new questions. Here we show that asymptotic symmetries correctly account for the entropy of the two dimensional black holes; on the other hand we recently showed that regularity at the horizon also do the job [5]. As for AdS_3 , this seems to suggest that there is some correspondence between states at infinity and on the horizon in 2D systems. But unlike from AdS_3 , the central charge here is equal to the one calculated using near horizon symmetries, and in fact the AdS_2 black hole simply is a foliation of AdS_2 by a different choice of time coordinate.

³These are described in [13, 14, 15, 16, 17, 18].

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